Dynamical systems method for solving linear ill-posed problems

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Abstract

Various versions of the Dynamical Systems Method (DSM) are proposed for solving linear ill-posed problems with bounded and unbounded operators. Convergence of the proposed methods is proved. Some new results concerning discrepancy principle for choosing regularization parameter are obtained.

1 Introduction

In this paper part of the results from the author's invited talk at ICMAAS06 are presented. This part deals with linear ill-posed problems.

Some of the results in this paper are taken from the papers of the author cited in the bibliography and many results are new. The DSM method is developed in [7], [8]-[28]. The discrepancy principle, discussed earlier in [5], has been studied recently in [8]-[10] and in [7].

Consider an equation

$$Au - f = 0, \tag{1.1}$$

where A is an operator in a Banach space X. If A is a homeomorphism of X onto X (i.e., continuous injective and surjective map in X which has a continuous inverse map) then problem (1.1) is called well-posed in the Hadamard's sense. Otherwise it is called ill-posed.

The Dynamical Systems Method (DSM) for solving equation (1.1) consists of solving the Cauchy problem

$$\dot{u} = \Phi(t, u), \qquad u(0) = u_0; \qquad \dot{u} = \frac{du}{dt}, \tag{1.2}$$

which we call a dynamical system, where Φ is chosen so that

$$\exists! u(t) \quad \forall t \geq 0; \qquad \exists u(\infty); \qquad A(u(\infty)) = f. \tag{1.3}$$

Here $u(\infty) := \lim_{t\to\infty} u(t)$. We do not assume that the solution to (1.1) is unique, but we do assume that there exists a solution to (1.1). If (1.3) holds, then we say that DSM is justified for solving equation (1.1).

There is a large body of literature on solving ill-posed problems (see e.g., [2], [7], [29] and references therein). Variational regularization, iterative regularization, quasisolutions and quasiinversion are some of the methods for stable solution of ill-posed problems discussed in the literature. In this paper several new methods for stable solutions of linear ill-posed problems is discussed. They are based on the dynamical systems method (DSM). We describe the DSM for solving ill-posed problems. This method was developed relatively recently as a method for solving a wide variety of linear and nonlinear ill-posed problems [8]-[28] although it was proposed already in [1] for solving well-posed problems.

Because of the space limitations we will not discuss solving nonlinear ill-posed problems by the DSM in this paper, and refer the reader to [7], [16].

Here we describe a new version of DSM for solving linear ill-posed problems. There are many practical problems of this type. We only mentioned solving ill-conditioned linear algebraic systems and Fredholm equations of the first kind. The novel points in our results are not only the method of solving these problems by DSM but also the applicability of the method to unbounded operators. In the literature a widely-discussed method for solving ill-posed problem (1.1) is the method of variational regularization introduced by Phillips ([6]) and studied by Tikhonov ([29]), Morozov ([5]), Ivanov ([2]), and many other authors under the assumption that the operator A in (1.1) is a linear bounded operator. There are some results on regularization of unbounded operators also (e.g., [4], [28]). The variational regularization method for stable solution of (1.1) consists of solving the problem

$$F(u) := \|Au - f_\delta\|^2 + a\|u\|^2 = \min, \tag{1.4}$$

where $a > 0$ is a constant, called a regularization parameter, f_δ are the "noisy data", i.e., an element which satisfies the inequality $\|f_\delta - f\| \leq \delta$ and which is given together with the "noise level" $\delta > 0$, while the exact data f are not known. The stable solution to (1.1) is an element u_δ such that $\lim_{\delta\to 0} \|u_\delta - y\| = 0$, where $Ay = f$ and y is the unique minimal-norm solution of the linear equation (1.1). If X is a Hilbert space H, which we assume below, then the minimal-norm solution is the solution which is orthogonal to the null space N of A, $N = N(A) = \{u : Au = 0\}$. If the linear operator A in (1.1) is unbounded, then we assume that it is closed and densely defined, so that its adjoint A^* is densely defined and closed (see, e.g., [3]).

If A is bounded, then the necessary and sufficient condition for the minimizer of the quadratic functional (1.4) is the Euler's equation

$$T_a u = A^* f_\delta, \quad T_a := T + aI, \quad T := A^* A, \tag{1.5}$$

where I is the identity operator and $T \geq 0$ is a selfadjoint operator. Equation (1.5) has a unique solution $u_{a,\delta} = T_a^{-1} A^* f_\delta$. One can choose $a = a(\delta)$ so that $\lim_{\delta \to 0} a(\delta) = 0$ and $u_\delta := u_{a(\delta),\delta}$ is a stable solution to (1.1):

$$\lim_{\delta \to 0} \|u_\delta - y\| = 0, \tag{1.6}$$

where $Ay = 0$, $y \perp N$. There are a priori choices for choosing $a(\delta)$ and a posteriori ones. An a priori choice is based on the estimate:

$$\|T_a^{-1} A^* f_\delta - y\| \leq \|T_a^{-1} A^* (f_\delta - f)\| + \|T_a^{-1} A^* f - y\| \leq \frac{\delta}{2\sqrt{a}} + \eta(a), \tag{1.7}$$

where

$$\eta^2(a) = \|T_a^{-1} T y - y\|^2 = a^2 \int_0^\infty \frac{d(E_s y, y)}{(s + a)^2} \to \|P_N y\|^2 \text{ as } a \to 0, \tag{1.8}$$

and P_N is the orthoprojector onto N. Since we assume that $y \perp N$, equation (1.8) implies

$$\lim_{a \to 0} \eta(a) = 0. \tag{1.9}$$

The term $\frac{\delta}{2\sqrt{a}}$ in (1.7) appears due to the estimate:

$$\|T_a^{-1} A^*\| = \|A^* Q_a^{-1}\| = \|U Q^{\frac{1}{2}} Q_a^{-1}\| \leq \|Q^{\frac{1}{2}} Q_a^{-1}\| = \sup_{s \geq 0} \frac{\sqrt{s}}{s + a} = \frac{1}{2\sqrt{a}}. \tag{1.10}$$

Here $Q := AA^*$, U is a partial isometry, $Q_a := Q + aI$, we have used the formula

$$T_a^{-1} A^* = A^* Q_a^{-1}, \tag{1.11}$$

the polar decomposition: $A^* = U Q^{\frac{1}{2}}$, and the spectral theorem for the selfadjoint operator Q, namely $\|g(Q)\| = \sup_{s \geq 0} |g(s)|$. Formula (1.11) is obvious if A is bounded: multiply (1.11) by T_a from the left and then by Q_a from the right, and get $A^*(AA^* + aI) = (A^* A + aI) A^*$, which is an obvious identity. Since the operators Q_a and T_a are boundedly invertible, one may reverse the steps and get (1.11). Thus a priori choices of $a(\delta)$, which imply (1.6), are:

$$\lim_{\delta \to 0} \frac{\delta}{\sqrt{a(\delta)}} = 0, \qquad \lim_{\delta \to 0} a(\delta) = 0. \tag{1.12}$$

There are many functions $a(\delta)$ satisfying (1.12). One can find an optimal value $a(\delta)$ by minimizing the right-hand side of (1.7) with respect to a. Alternatively, one may calculate $a(\delta)$ by solving the equation $\delta = 2\sqrt{a(\delta)}\eta(a)$ for a for a fixed small $\delta > 0$.

If A is closed, densely defined in H, unbounded, and $a = \text{const} > 0$, then the author has proved that the operator $T_a^{-1} A^*$, with the domain $D(A^*)$, is closable, its closure, denoted again $T_a^{-1} A^*$, is a bounded operator defined on all of H, $\|T_a^{-1} A^*\| \leq \frac{1}{2\sqrt{a}}$, and (1.1) holds. For convenience of the reader let us sketch the proof of these claims. To

check that $T_a^{-1}A^*$ is closable, one takes $h_n \in D(T_a^{-1}A^*) = D(A^*)$, such that $h_n \to 0$ and $T_a^{-1}A^*h_n \to g$ as $n \to \infty$, and checks that $g = 0$. Indeed, let $u \in H$ be arbitrary. Then

$$(g, u) = \lim_{n\to\infty} (T_a^{-1}A^*h_n, u) = \lim_{n\to\infty} (h_n, AT_a^{-1}u) = 0. \qquad (1.13)$$

Since u is arbitrary, this implies $g = 0$, as claimed. Note that $T_a^{-1}u \in D(A)$, so that the above calculation is justified. If one drops the index n and the lim in (1.13), then one can see that the adjoint to the closure of $T_a^{-1}A^*$ is the operator AT_a^{-1}, defined on all of H and bounded

$$\|AT_a^{-1}\| = \|UT^{\frac{1}{2}}T_a^{-1}\| \le \frac{1}{2\sqrt{a}}.$$

Since $\|A^*\| = \|A\|$, one gets $\|T_a^{-1}A^*\| \le \frac{1}{2\sqrt{a}}$. Finally formula (1.11) can be proved for unbounded closed, densely define operator A as above, if one checks that the operator A^*AA^* is densely defined. This operator is densely defined because the operator $A^*AA^*A = T^2$ is densely defined if T is, and $D(T^2) \subset D(A^*AA^*)$.

Let us now describe an a posteriori choice of $a(\delta)$ which implies (1.6) and which is called a discrepancy principle. This principle was discussed in [5], [7]. It consists in finding $a(\delta)$ from the equation

$$\|Au_{a,\delta} - f_\delta\| = C\delta, \qquad 1 < C < 2, \qquad (1.14)$$

where $C = const$, $\|F_\delta\| > C\delta$ and $u_{a,\delta} = T_a^{-1}A^*f_\delta$. One can prove (e.e. see [7]) that the equation (1.14) for a small fixed $\delta > 0$ has a unique solution $a(\delta)$, $\lim_{\delta\to 0} a(\delta) = 0$, and $u_\delta = u_{a(\delta),\delta}$ satisfies (1.6), i.e. u_δ is a stable solution to (1.1). To prove these claims one writes (1.14) as

$$C^2\delta^2 = \|[AT_a^{-1}A^* - I]f_\delta\|^2 = \|[QQ_a^{-1} - I]f_\delta\|^2 = \int_0^\infty \frac{a^2 d(E_\delta f_\delta, f_\delta)}{(s+a)^2} := h(a, \delta), \quad (1.15)$$

and takes into account that $h(a, \delta)$ is, for a fixed $\delta > 0$, a continuous monotone function of a, $h(\infty, \delta) = \|f_\delta\|^2 > C^2\delta^2$ and $h(+0, \delta) = \|P_{N^*}f_\delta\| \le \delta^2$, so that there exists a unique $a = a(\delta)$ such that $h(a(\delta), \delta) = C^2\delta^2$. Here we have denoted $N^* := N(A^*)$, used the obvious relation $N(Q) = N(A^*)$, the inequality $\|P_{N^*}f_\delta\| \le \|P_{N^*}(f_\delta - f)\| + \|P_{N^*}f\| \le \|f_\delta - f\| = \delta$, and the relation $P_{N^*}f = 0$. This last relation follows from the formulas $f \in R(A)$ and $R(A) \perp N^*$.

Let us now check that if $a(\delta)$ solves (1.5) then $u_\delta = a_{a(\delta),\delta}$ satisfies (1.6). One has $F(u_\delta) \le F(y)$, so

$$\|Au_\delta - f_\delta\|^2 + a(\delta)\|u_\delta\|^2 \le \delta^2 + a(\delta)\|y\|^2. \qquad (1.16)$$

Since $\|Au_\delta - f_\delta\|^2 = C^2\delta^2 > \delta^2$ and $a(\delta) > 0$, one gets

$$\|u_\delta\| \le \|y\|. \qquad (1.17)$$

Therefore one can select a weakly convergent sequence $u_n = u_{\delta_n}$, $u_n \rightharpoonup u$, as $n \to \infty$. Let us prove that $u = y$ and $\lim_{n\to\infty}\|u_n - y\| = 0$. Since this holds for any subsequence,

it follows that (1.6) holds. To prove that $u = y$ note that (1.17) implies $\|u\| \leq \|y\|$, and that u solves (1.1). Since the minimal-norm solution to (1.1) is unique, it follows that $u = y$. To check that u solves (1.1) we note that $\lim_{\delta \to 0} \|Au_\delta - f\| = 0$ as follows from (1.16) because $\lim_{\delta \to 0} a(\delta) = 0$. The relations $u_\delta \rightharpoonup u$ and $\|Au_\delta - f\| \to 0$ as $\delta \to 0$ imply $Au = f$ and $\lim_{\delta \to 0} \|u_\delta - u\| = 0$. Let us first check that $Au = f$. One has

$$(f, g) = \lim_{\delta \to 0}(Au_\delta, g) = (u, A^*g) \qquad \forall g \in D(A^*).$$

Thus $u \in D(A)$ and $Au = f$, as claimed. As we proved above, this implies that $u = y$. Therefore $u_\delta \rightharpoonup y$, $\|u_\delta\| \leq \|y\|$. This implies that $\lim_{\delta \to 0} \|u_\delta - y\| = 0$. Indeed,

$$\|u_\delta - y\|^2 = \|u_\delta\|^2 + \|u\|^2 - 2Re(u_\delta, y) \leq 2\|y\|^2 - 2Re(u_\delta, y) \to 0 \quad \text{as } \delta \to 0. \quad (1.18)$$

Thus the relation (1.6) is proved for the choice of $a(\delta)$ by the discrepancy principle.

The drawback of the a priori choice if $a(\delta)$ is that it is nonunique and although it guarantees convergence (1.16), the error of the method can be large if $\delta > 0$ is fixed. The drawback of the discrepancy principle is the necessity of solving the nonlinear equation (1.14) and also a possible large error for a fixed δ.

In Section 2 we discuss the DSM for solving linear equations (1.1).

2 DSM for solving linear problems

We assume first that the linear closed densely defined in H operator in (1.1) is selfadjoint, $A = A^*$. This is not a restriction: every solvable linear equation (1.1) is equivalent to an equation $Tu = A^*f$, where $T = T^* = A^*A$. Indeed, if $Au = f$, then applying A^* and assuming $f \in D(A^*)$, one gets $Tu = A^*f$. Conversely, if $Tu = A^*f$, and $f = Ay$, then $Tu = Ty$. Multiply the equation $0 = T(u - y)$ by $u - y$ and get $0 = (A^*A(u-y), u-y) = \|Au - Ay\|^2$. Thus $Au = Ay = f$. If A is bounded, then $f \in D(A^*)$ for any $f \in H$. If A is unbounded, then $D(A^*)$ is a linear dense subset of H. In this case, if $f \notin D(A^*)$, then we define the solution of equation $Tu = A^*f$ by the formula $u = \lim_{a \to 0} T_a^{-1}A^*f$. As we have proved in Section 1, for any $f \in R(A)$ this limit exists and equals to the minimal-norm solution y: $\lim_{a \to 0} T_a^{-1}A^*Au = y$ if $Au = f$. This is true because $\lim_{a \to 0} T_a^{-1}Tu = u - P_N u = y$.

The DSM for solving equation (1.1) with a linear selfadjoint operator can be constructed as follows. Consider the problem

$$\dot{u}_a = i(A + ia)u_a - if, \qquad u(0) = 0; \qquad \dot{u} = \frac{du}{dt}, \qquad (2.1)$$

where $a = const > 0$. Our first result is formulated as Theorem 1.

Theorem 1. *If $Ay = f$ and $y \perp N$, then*

$$\lim_{a \to 0} \lim_{t \to \infty} u_a(t) = y. \qquad (2.2)$$

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Our second result shows that the method, based on Theorem 1, gives a stable solution of the equation $Au = f$. Assume that $\|f_\delta - f\| \leq \delta$, and let $u_{a,\delta}(t)$ be the solution to (2.1) with f_δ in place of f.

Theorem 2. *There exist $t = t_\delta$, $\lim_{\delta \to 0} t_\delta = \infty$, and $a = a(\delta)$, $\lim_{\delta \to 0} a(\delta) = 0$, such that $u_\delta := u_{a(\delta),\delta}(t_\delta)$ satisfies (1.6).*

We will discuss the ways to choose $a(\delta)$ and t_δ after the proofs of these theorems are given.

From the numerical point of view if one integrates problem (2.1) with the exact data f on the interval $0 \leq t \leq T$, and T is fixed, then one is interested in choosing such $a = a(T)$ that $\lim_{T \to \infty} \|u_{a(T)}(T) - y\| = 0$. We will give such a choice of $a(T)$.

Before we start proving these two theorems, let us explain the ideas of the proof. Suppose B is a linear operator and its inverse B^{-1} exists and is bounded.

Finally, assume that
$$\lim_{t \to \infty} \|e^{Bt}\| = 0. \tag{2.3}$$

This will happen, for example, if $Re\, B \leq -c$, $c > 0$ is a constant. Under these assumptions one has
$$\int_0^t e^{Bs} ds = B^{-1}\left(e^{Bt} - I\right), \quad -\lim_{t \to \infty} \int_0^t e^{Bs} ds = B^{-1}. \tag{2.4}$$

The operator $\int_0^t e^{Bs} ds$ solves the problem
$$\dot{W} = BW + I, \qquad W(0) = 0, \tag{2.5}$$

where I is the identity operator. If $\lim_{t \to \infty} \|e^{Bt}\| = 0$, then
$$-\lim_{t \to \infty} W(t) = B^{-1}. \tag{2.6}$$

The basic idea of the DSM is the representation of the inverse operator as the limit as $t \to \infty$ of the solution to the Cauchy problem (2.5).

Let us prove Theorem 1.

Proof of Theorem 1. The solution to (2.1) is
$$u_a(t) = \int_0^t e^{i(A+ia)(t-s)}(if)ds = [i(A + ia)]^{-1}\left(e^{i(A+ia)t} - I\right)(-if).$$

Since $\|e^{i(A+ia)t}\| = e^{-at} \to 0$ as $t \to \infty$, one gets
$$\lim_{t \to \infty} u_a(t) = (A + ia)^{-1}f. \tag{2.7}$$

Since $f = Ay$, one has
$$\eta(a) := \|(A + ia)^{-1}Ay - y\| = a\|(A + ia)^{-1}y\| \to 0 \quad \text{as } a \to 0. \tag{2.8}$$

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The last relation follows from the assumption $y \perp N$. Indeed, by the spectral theorem one has:

$$
\lim_{a \to 0} \eta^2(a) = \lim_{a \to 0} a^2 \|(A + ia)^{-1} y\|^2
$$

$$
= \lim_{a \to 0} \int_{-\infty}^{\infty} \frac{a^2}{s^2 + a^2} d(E_s y, y) = \|(E_0 - E_{0-0}) y\|^2 = \|P_N y\|^2 = 0.
$$

Theorem 1 is proved. □

Remark 1. *In a numerical implementation of Theorem 1 one chooses τ, and $a = a(\tau)$, and integrates (2.1) on the interval $[0, \tau]$. One chooses τ so that*

$$
\lim_{\tau \to \infty} \|u_{a(\tau)} - y\| = 0. \tag{2.9}
$$

In order to choose $a(\tau)$, note that $\|u_a(t) - u_a(\infty)\| \leq \frac{e^{-at}}{a}$, as follows from the derivation of (2.7). Therefore, by Theorem 1, the relation (2.9) holds if

$$
\lim_{\tau \to \infty} \frac{e^{-a(\tau)\tau}}{a(\tau)} = 0 \qquad \lim_{\tau \to \infty} a(\tau) = 0. \tag{2.10}
$$

For example, one may take $a(\tau) = \tau^{-\gamma}$, where $0 < \gamma < 1$ is a constant.

Proof of Theorem 2. Let us start with the formula:

$$
u_{a,\delta}(t) = \int_0^t e^{i(A+ia)(t-s)}(-i f_\delta) ds = [i(A + ia)]^{-1} \left(e^{i(A+ia)t} - I \right) (i f_\delta). \tag{2.11}
$$

Thus

$$
\mathcal{E} := \|u_{a,\delta}(t) - y\| \leq \|u_{a,\delta}(t) - u_a(t)\| + \|u_a(t) - y\|. \tag{2.12}
$$

One has

$$
\|u_{a,\delta}(t) - u_a(t)\| \leq \frac{\|f_\delta - f\|}{a} \|I - e^{iAt - at}\| \leq \frac{2\delta}{a}, \tag{2.13}
$$

and

$$
\|u_a(t) - y\| \leq \frac{e^{-at}}{a} \|f\| + \eta(a), \tag{2.14}
$$

where $\eta(a)$ is defined in (2.8), $\lim_{a \to 0} \eta(a) = 0$. Since $\|f\| \leq \|f_\delta\| + \delta \leq c$, one obtains from (2.12)-(2.14):

$$
\lim_{\delta \to 0} \mathcal{E} = 0 \tag{2.15}
$$

provided that $t = t_\delta$, $a = a(\delta)$ and

$$
\lim_{\delta \to 0} t_\delta = \infty, \qquad \lim_{\delta \to 0} a(\delta) = 0, \qquad \lim_{\delta \to 0} \frac{e^{-a(\delta)t_\delta}}{a(\delta)} = 0, \qquad \lim_{\delta \to 0} \frac{\delta}{a(\delta)} = 0. \tag{2.16}
$$

Theorem 2 is proved. □

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Remark 2. *There are many choices of t_δ and $a(\delta)$ satisfying relations (2.16). If one has an estimate of the rate of decay of $\eta(a)$ as $a \to 0$, then one may obtain some rate of convergence of $\mathcal{E}$ to zero as $\delta \to 0$. However, it is impossible, in general, to get a rate of decay of $\eta(a)$ as $a \to 0$ without additional assumptions on the data f or on the solution y. A typical assumption is $y = Az$, that is, $y \in R(A)$. If fractional powers of A are defined (which is the case when $A \geq 0$, for example) then one may assume $y = A^\gamma z$, $\gamma > 0$. Let us show how to get the rate of decay of $\eta(a)$ under such assumptions. Assume, for example, that $y = Az$. Then*

$$\eta^2(a) = \int_{-\infty}^{\infty} \frac{a^2 s^2}{a^2 + s^2} d(E_s z, z) \leq a^2 \|z\|^2, \tag{2.17}$$

and the error bound is

$$\mathcal{E} \leq \frac{2\delta}{a} + c_1 \frac{e^{-at}}{a} + c_2 a, \qquad c_2 = \|z\|, \quad c_1 = \|f_\delta\| + \delta. \tag{2.18}$$

Choose, for example, $a = \delta^\gamma$, $0 < \gamma < 1$, and $t_\delta = \delta^{-\mu}$, $\mu > \gamma$. Then $\lim_{\delta \to 0} \frac{e^{-a(\delta)t_\delta}}{a(\delta)} = 0$ and (2.15) holds with the rate δ^ν, $\nu = min(1-\gamma, \gamma)$. If $\gamma = \frac{1}{2}$ then $\max_{0<\gamma<1} min(1-\gamma, \gamma)$ is equal to $\frac{1}{2}$, and for $\gamma = \frac{1}{2}$ one gets $\mathcal{E} = O(\delta^{1/2})$ if $t_\delta = \delta^{-\mu}$, $\mu > \frac{1}{2}$.

3 Second version of the DSM

Consider problem (2.1) with $a = a(t)$. Let us assume that $a(t) > 0$ is a continuous function monotonically decaying to zero as $t \to \infty$ and

$$0 < a(t) \searrow 0, \qquad a' + a^2 \in L^1(0, \infty), \qquad \int_0^\infty a(s)ds = \infty. \tag{3.1}$$

The solution to this problem is

$$u(t) = \int_0^t e^{iA(t-s) - \int_s^t a(p)dp} ds(-if). \tag{3.2}$$

Theorem 3. *Under the above assumptions one has*

$$\lim_{t \to \infty} \|u(t) - y\| = 0. \tag{3.3}$$

Proof of Theorem 3. Since $f = Ay$, one gets, integrating by parts,

$$u(t) = e^{iAt} e^{-iAs - \int_s^t adp} y\big|_0^t - \int_0^t e^{iA(t-s)} a(s) e^{-\int_s^t a(p)dp} ds y.$$

Thus

$$u(t) = y - e^{iAt - \int_0^t adp} y - \int_0^t e^{iA(t-s)} a(s) e^{-\int_s^t adp} ds y, \tag{3.4}$$

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and
$$\|u(t) - y\| \le e^{-\int_0^t a\,dp}\|y\| + \|\int_0^t e^{iA(t-s)}a(s)e^{-\int_s^t a\,dp}ds\,y\| := J_1 + J_2. \qquad (3.5)$$

By the last assumption (3.1) one gets
$$\lim_{t\to\infty} J_1 = 0 \qquad (3.6)$$

We now prove that
$$\lim_{t\to\infty} J_2 = 0. \qquad (3.7)$$

Using the spectral theorem, one gets
$$J_2^2 = \int_{-\infty}^{\infty} d(E_\lambda y, y)\left|\int_0^t e^{i\lambda(t-s)}a(s)e^{-\int_s^t a\,dp}ds\right|^2. \qquad (3.8)$$

Let us prove that
$$\lim_{t\to\infty}\int_0^t e^{i\lambda(t-s)}a(s)e^{-\int_s^t a\,dp}ds = 0, \qquad \forall\lambda \ne 0. \qquad (3.9)$$

From (3.8), (3.9) and the assumption $y \perp N$, the conclusion (3.7) follows.

Let us verify (3.9). Integrating by parts one gets:
$$J_3 := \int_0^t e^{i\lambda(t-s)}a(s)e^{-\int_s^t a\,dp}ds = \frac{e^{i\lambda(t-s)}}{-i\lambda}a(s)e^{-\int_s^t a\,dp}\Big|_0^t + \frac{1}{i\lambda}\int_0^t e^{i\lambda(t-s)}[a'(s)+a^2(s)]e^{-\int_s^t a(p)dp}ds. \qquad (3.10)$$

Thus
$$J_3 = \frac{a(t)}{-i\lambda} + \frac{e^{i\lambda t - \int_0^t a\,dp}a(0)}{i\lambda} + J_4, \qquad \lambda \ne 0, \qquad (3.11)$$

where J_4 denotes the last integral in (3.10). The first two terms in (3.11) tend to zero as $t \to \infty$ because of the assumptions about $a(t)$.

Assumptions (3.1) imply that
$$\lim_{t\to\infty} J_4 = 0. \qquad (3.12)$$

Thus, Theorem 3 is proved. $\qquad\qquad\square$

Remark 3. *For example, the function* $a(t) = \frac{c_0}{(c_1+t)^b}$, *where* $c_0, c_1 > 0$ *are arbitrary constants and* $b \in (\frac{1}{2}, 1)$ *is a constant, satisfies assumptions* (3.1).

It is interesting to check numerically the efficiency of the algorithm based on Theorem 3.

Let us prove that Theorem 3 yields a stable solution to equation (1.1).

Theorem 4. *There exists a stopping time t_δ, $\lim_{\delta \to 0} t_\delta = \infty$, such that (1.6) holds with $u_\delta = u_\delta(t_\delta)$, where $u_\delta(t)$ is the solution to problem (2.1) with $a = a(t)$ and f_δ in place of f, $\|f_\delta - f\| \leq \delta$.*

Proof of Theorem 4. One has

$$\|u_\delta(t) - y\| \leq \|u_\delta(t) - u(t)\| + \|u(t) - y\|, \tag{3.13}$$

where $u(t)$ solves problem (2.1) with $a = a(t)$ and exact data. We have proved in Theorem 3 that

$$\lim_{t \to \infty} \|u(t) - y\| = 0. \tag{3.14}$$

We have

$$\|u_\delta(t) - u(t)\| \leq \int_0^t e^{-\int_s^t a(p)dp} ds \|f_\delta - f\| \leq \frac{\delta}{a(t)}. \tag{3.15}$$

Here the estimate $\int_0^t e^{-\int_s^t adp} ds \leq \frac{1}{a(t)}$ was used. This estimate is derived easily:

$$\int_0^t e^{-\int_s^t adp} ds \leq \frac{1}{a(t)} \int_0^t a(s) e^{-\int_s^t adp} dp = \frac{1}{a(t)} e^{-\int_s^t adp}\Big|_0^t = \frac{1}{a(t)}\left(1 - e^{-\int_0^t adp}\right) \leq \frac{1}{a(t)}.$$

Choose t_δ so that

$$\lim_{\delta \to 0} t_\delta = \infty, \qquad \lim_{\delta \to 0} \frac{\delta}{a(t_\delta)} = 0. \tag{3.16}$$

This is obviously possible to do.

Then (3.13)-(3.15) imply

$$\lim_{\delta \to 0} \|u_\delta(t_\delta) - y\| = 0. \tag{3.17}$$

Theorem 4 is proved. $\square$

4 Third version of DSM for solving equation (1.1)

$$\dot{u} = -u + T_{a(t)}^{-1} A^* f, \qquad u(0) = 0, \tag{4.1}$$

where $f = Ay$, $y \perp N$, and $a(t) > 0$ is a monotonically decaying continuous function such that $\lim_{t \to \infty} a(t) = 0$ and $\int_0^\infty a(t)dt = \infty$. We could take the initial condition $u(0) = u_0$ arbitrary. The contribution to the solution to problem (4.1) which comes from the initial condition u_0 is the term $u_0 e^{-t}$. It decays exponentially fast and our arguments do not depend on this term essentially. To simplify and shorten our argument we take $u_0 = 0$.

Theorem 5. $\lim_{t \to \infty} u(t) = y$.

Proof of Theorem 5. One has

$$u(t) = \int_0^t e^{-(t-s)} T_{a(s)}^{-1} A^* A y ds = \int_0^t e^{-(t-s)} y ds - \int_0^t e^{-(t-s)} a(s) T_{a(s)}^{-1} y ds.$$

Thus

$$\|u(t) - y\| \le e^{-t} \|y\| + \int_0^t e^{-(t-s)} a(s) \|T_{a(s)}^{-1} y\| ds. \tag{4.2}$$

One can easily check that if $b(s)$ is a continuous function on $[0, \infty)$ and $b(\infty) = \lim_{s \to \infty} b(s)$ exists, then

$$\lim_{t \to \infty} \int_0^t e^{-(t-s)} b(s) ds = b(\infty). \tag{4.3}$$

Thus, Theorem 5 will be proved if one checks that

$$\lim_{a \to 0} a \|T_a^{-1} y\| = 0 \qquad \forall y \in H. \tag{4.4}$$

To prove (4.4) one writes, using the spectral theorem:

$$a^2 \|T_a^{-1} y\|^2 = \int_0^\infty \frac{a^2}{(s+a)^2} d(E_s y, y) = \|P_N y\|^2 = 0. \tag{4.5}$$

Theorem 5 is proved. □

Let us prove that the DSM method (4.1) yields a stable solution to problem (1.1). Assume that f is replaced by f_δ, $\|f_\delta - f\| \le \delta$, in equation (4.1) and denote by $u_\delta(t)$ the corresponding solution. Then, using the estimate (3.13), one gets

$$\lim_{\delta \to 0} \|u_\delta(t_\delta) - y\| = 0, \tag{4.6}$$

provided that

$$\lim_{\delta \to 0} t_\delta = \infty, \qquad \lim_{\delta \to 0} \frac{\delta}{\sqrt{a(t_\delta)}} = 0. \tag{4.7}$$

To check the sufficiency of the second condition (4.7) for (4.6) to hold, one proceeds as follows:

$$\|u_\delta(t) - u(t)\| = \left\| \int_0^t e^{-(t-s)} T_{a(s)}^{-1} A^* (f_\delta - f) ds \right\|$$

$$\le \delta \int_0^t e^{-(t-s)} \frac{1}{2\sqrt{a(s)}} ds \le \frac{\delta}{2\sqrt{a(t)}}. \tag{4.8}$$

Here we have used the monotonicity of $a(t)$, which implies $a(t) \le a(s)$ if $t \ge s$, and the estimate $\|T_a^{-1} A^*\| \le \frac{1}{2\sqrt{a}}$, which was proved earlier.

Let us state the result we have proved.

Theorem 6. *If t_δ is chosen so that (4.7) holds, then the solution $u_\delta(t)$ to problem (4.1) with noisy data f_δ in place of f satisfies (4.6).*

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5 A new discrepancy principle

The usual discrepancy principle is described in Section 1. It requires solving nonlinear equation $\|Au_{a,\delta} - f\| = C\delta$, $C = const$, $1 < C < 2$, where $u_{a,\delta} = T_a^{-1}A^*f_\delta$. Thus one has to know the exact minimizer $u_{a,\delta}$ of the functional $F(u) = \|Au - f_\delta\|^2 + a\|u\|^2$, or the exact solution of the equation $T_a u = A^*f_\delta$.

In this Section we discuss the following question:

How does one formulate the discrepancy principle in the case when $u_{a,\delta}$ is not the exact solution of the minimization problem $F(u) = \min$, but an approximate solution.

Let us state the result.

Theorem 7. *Assume that A is a bounded linear operator in a Hilbert space H, that $f = Ay$, $y \perp N$, $\|f_\delta - f\| \leq \delta$, $\|f_\delta\| > C\delta$, $C = const$, $C \in (1,2)$, and $u_{a,\delta}$ is any element which satisfies the inequality:*

$$F(u_{a,\delta}) \leq m + (C^2 - 1 - b)\delta^2, \tag{5.1}$$

where

$$F(u) = \|Au - f_\delta\|^2 + a\|u\|^2, \qquad m = \inf_{u \in H} F(u), \tag{5.2}$$

$b = const > 0$ and $C^2 > 1 + b$.

Then equation

$$\|Au_{a,\delta} - f_\delta\| = C\delta \tag{5.3}$$

has a solution for any fixed $\delta > 0$, $\lim_{\delta \to 0} a(\delta) = 0$, and

$$\lim_{\delta \to 0} \|u_\delta - y\| = 0 \tag{5.4}$$

where $u_\delta = u_{a(\delta),\delta}$ and $a(\delta)$ solves (5.3).

Proof of Theorem 7. To prove the existence of a solution to (5.3) denote $\|Au_{a,\delta} - f_\delta\|$ by $h(\delta, a)$, check that $h(\delta, +0) < C\delta$, $h(\delta, \infty) > C\delta$ and note that $h(\delta, a)$ is a continuous function of a on the interval $(0, \infty)$. Then equation (5.3) has a solution $a = a(\delta)$. As $a \to \infty$ one has

$$a\|u_{a,\delta}\|^2 \leq F(u_{a,\delta}) \leq m + (c^2 - 1 - b)\delta^2 \leq F(0) + (C^2 - 1 - b)\delta^2$$

so, with $c := F(0) + (C^2 - 1 - b)\delta^2$, one obtains

$$\|u_{a,\delta}\| \leq \frac{c}{\sqrt{a}}, \qquad a \to \infty.$$

Thus

$$h(\delta, \infty) = \|A0 - f_\delta\| = \|f_\delta\| > C\delta. \tag{5.5}$$

As $a \to 0$, one has

$$h^2(\delta, a) \leq F(u_{a,\delta}) \leq m + (C^2 - 1 - b)\delta^2 \leq F(y) + (C^2 - 1 - b)\delta^2.$$

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One has $F(y) = \delta^2 + a\|y\|^2$, so $h^2(\delta, a) \leq (C^2 - b)\delta^2 + a\|y\|^2$, and

$$h(\delta, +0) \leq (C^2 - b)^{\frac{1}{2}}\delta < C\delta. \tag{5.6}$$

Finally, the continuity of $h(\delta, a)$ with respect to a for any fixed $\delta > 0$ follows from the continuity of the bounded operator a and the assumed continuity of $u_{a,\delta}$ with respect to a. Thus, the existence of a solution $a = a(\delta) > 0$ of equation (5.3) is proved. One takes a solution for which $\lim_{\delta \to 0} a(\delta)$. Such solution exists because $h(\delta, +0)$ and $m = m(\delta, a)$ tend to zero as $\delta \to 0$ and $a \to 0$.

Let us prove (5.4). One has

$$F(u_\delta) = \|Au_\delta - f_\delta\|^2 + a(\delta)\|u_\delta\|^2 \leq \|Ay - f_\delta\|^2 + a(\delta)\|y\|^2 \leq \delta^2 + a(\delta)\|y\|^2. \tag{5.7}$$

Since $\|Au_\delta - f_\delta\| = C\delta$, $C > 1$ it follows from (5.7) that

$$\|u_\delta\| \leq \|y\|. \tag{5.8}$$

Thus one may assume that $u_\delta \rightharpoonup u$ as $\delta \to 0$.

Let us prove that $Au = f$. First we observe that

$$\|Au_\delta - f\| \leq \|Au_\delta - f_\delta\| + \|f_\delta - f\| \leq C\delta + \delta,$$

so

$$\lim_{\delta \to 0} \|Au_\delta - f\| = 0 \tag{5.9}$$

Secondly, for any $v \in H$ we have

$$(f - Au, v) = \lim_{\delta \to 0}(Au_\delta - Au, v) = \lim_{\delta \to 0}(u_\delta - u, A^*v) = 0, \tag{5.10}$$

because $u_\delta \rightharpoonup u$. Since v is arbitrary, one concludes from (5.10) that $Au = f$. From (5.8) it follows that $\|u\| \leq \|y\|$. Since the minimal-norm solution to equation $Au = f$ is unique, one obtains $u = y$. Thus $u_\delta \rightharpoonup y$, $\|u_\delta\| \leq \|y\|$. This implies (5.4) as follows from (1.18).

Theorem 7 is proved. □

6 Discrepancy principle does not yield convergence uniformly with respect to the data

In this Section we make the following assumption.

Assumption A): *A is a linear bounded operator in a Hilbert space H, $N := N(A) = N(A^*) := N^* = \{0\}$, A^{-1} is unbounded, $Ay = f$, $\|f_\delta - f\| \leq \delta$, $\|f_\delta\| > \delta$.*

Let $a = a(\delta)$ be chosen by the discrepancy principle,

$$\|Au_{a,\delta} - f_\delta\| = C\delta, \qquad u_\delta = u_{a(\delta),\delta} = T_{a(\delta)}^{-1} A^* f_\delta. \tag{6.1}$$

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Consider the set S_δ:

$$S_\delta := \{v : \|Av - f_\delta\| \leq \delta\}.$$

We are interested in the following question: *given $\{f_\delta, \delta\}_{\delta \in (0,\delta_0)}$, where $\delta_0 > 0$ is a small number, and assuming that $a(\delta)$ is the solution to (6.1), can one guarantee uniformity with respect to the data f convergence?*

In other words, is it true that

$$\lim_{\delta \to 0} \sup_{v \in S_\delta} \|u_\delta - v\| = 0. \tag{6.2}$$

The answer is no.

Theorem 8. *There exist f_δ such that*

$$\lim_{\delta \to 0} \sup_{v \in S_\delta} \|u_\delta - v\| \geq c > 0, \qquad c = const. \tag{6.3}$$

Proof of Theorem 8. Denote $T_a^{-1}A^* := G$, $u_\delta = Gf_\delta$, $\|G\| = \frac{1}{2\sqrt{a}}$. We have proved in (1.10) that $\|G\| \leq \frac{1}{2\sqrt{a}}$, but the equality sign holds because in (1.10) U is unitary under **Assumptions** A). Thus, one can find an element $p = p_a$, $\|p\| = \frac{\delta}{2}$, such that

$$\|Gp\| \geq \frac{1}{2}\|G\| \ \|p\| = \frac{\delta}{8\sqrt{a}}. \tag{6.4}$$

Assumptions A) imply that the ranges $R(A)$ and $R(T)$ are dense in H. Thus one can find an element $z = z_{a,\delta}$ such that

$$\|f_\delta - AT^b z - p\| \leq \frac{\delta}{8}, \qquad b \in (0,1), \ b = const. \tag{6.5}$$

For any v one has

$$\|Gf_\delta - v\| \leq \|Gf_\delta - GAv\| + \|GAv - v\|. \tag{6.6}$$

Take $v = T^b z$ and let $M > 0$ be an arbitrary large fixed a_n constant. Then

$$\lim_{\delta \to 0} \sup_{v \in S_\delta, v = B^b z, \|z\| \leq m} \|GAv - v\| = 0, \tag{6.7}$$

because

$$\|GAv - v\| = \|T_a^{-1}Tv - v\| = \| - aT^{-1}T^b z\| = a \sup_{s \geq 0} \frac{s^b}{s + a} = ca^b, \tag{6.8}$$

where $c = b^b(1 - b)^{1-b}$.

From (6.7) and (6.6) one sees that

$$\lim_{\delta \to 0} \sup_{v \in S_\delta, v = T^b z, \|z\| \leq M} \|Gf_\delta - v\| = 0. \tag{6.9}$$

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if and only if

$$\lim_{\delta \to 0} \sup_{v \in S_\delta, v = T^b z, \|z\| \le M} \|Gf_\delta - GAv\| = 0. \tag{6.10}$$

Take $z = z_a$, $v = T^b z$. Then

$$\|f_\delta - Av\| \le \|p\| + \|f_\delta - AB^b z - p\| \le \frac{\delta}{2} + \frac{\delta}{8} = \frac{5\delta}{8} \le \delta, \tag{6.11}$$

and, using (6.5), one gets:

$$\|Gf_\delta - GAv\| = \|G(f_\delta - Av - p) + Gp\| \ge \|Gp\| - \|G\|\frac{\delta}{8} \ge \frac{\delta}{\sqrt{a}}\left(\frac{1}{8} - \frac{1}{16}\right) = \frac{\delta}{16\sqrt{a}}. \tag{6.12}$$

If $\frac{\delta}{\sqrt{a}} \ge c > 0$, then, according to (6.12), (6.10) fails.

Let us find f_δ such that for $a = a(\delta)$, defined by the discrepancy principle, one has $\frac{\delta}{\sqrt{a}} \ge c > 0$. This will complete the proof of Theorem 8. Let us assume for simplicity that $A = A^* > 0$ is compact. Then $T = A^*A = A^2$, and equation (6.1) becomes

$$C^2\delta^2 = \|[A(A^2 + a)^{-1}A - I]f_\delta\|^2 = \sum_{j=1}^{\infty}\left[\frac{\lambda_j^2}{\lambda_j^2 + a} - 1\right]^2 |f_{\delta j}|^2 = \sum_{j=1}^{\infty}\frac{a^2|f_{\delta j}|^2}{(\lambda_j + a)^2}. \tag{6.13}$$

Here λ_j are the eigenvalues of A^2, $f_{\delta j} = (f_\delta, \varphi_j)$, $A^2\varphi_j = \lambda_j\varphi_j$, $\|\varphi_j\| = 1$. Assume, for example, that $\lambda_j = \frac{1}{j}$ and $|f_{\delta j}|^2 = \frac{1}{j^2}$. Then (6.13) becomes

$$C^2\frac{\delta^2}{a^2} = \sum_{j=1}^{\infty}\frac{j^{-2}}{(j^{-1} + a)^2}. \tag{6.14}$$

Note that

$$I_1 := \int_1^\infty \frac{x^{-2}}{(x^{-1} + a)^2}dx$$
$$= \int_0^1 \frac{ds}{(s + a)^2} = -(s + a)^{-1}\Big|_0^1 = a^{-1}[1 - \frac{a}{a + 2}] = \frac{1}{a}[1 + O(a)], \quad a \to 0$$

This and (6.14) imply $\frac{\delta}{\sqrt{a}} \ge c > 0$ as $\delta \to 0$. Theorem 8 is proved. $\qquad\square$

7 Iterative processes for solving equation (1.1)

In this Section convergent iterative processes for solving equation (1.1) are constructed in the case when A is a closed, densely defined in H, unbounded operator. Consider the process

$$u_{n+1} = Bu_n + T_a^{-1}A^*f, \quad u_1 = u_1, \quad u_1 \perp N, \quad B := aT_a^{-1}, \tag{7.1}$$

where $a = const > 0$ and the initial element u_1 is arbitrary in the subspace $N^\perp$, $N := N(A) = N(T)$, $T = A^*A$, $T_a = T + aI$. Note that $B \ge 0$, $\|B\| \le 1$.

Theorem 9. *Under the above assumptions one has*

$$\lim_{n\to\infty} \|u_n - y\| = 0. \tag{7.2}$$

Proof of Theorem 9. Let $w_n = u_n - y$. Then

$$w_{n+1} = Bw_n = B^n w, \qquad w := u_1 - y, \qquad w \perp N. \tag{7.3}$$

Let us prove that

$$\lim_{n\to\infty} \|B^n w\| = 0. \tag{7.4}$$

If (7.4) is verified, then Theorem 9 is proved. We have

$$\|B^n w\|^2 = \int_0^\infty \frac{a^{2n}}{(a+s)^{2n}} d(E_s w, w) = \int_{s>b} + \int_{0\le s\le b} := J_1 + J_2, \tag{7.5}$$

where E_s is the resolution of the identity corresponding to the operator $T \ge 0$, and $b > 0$ is a small number which will be chosen later. For any fixed $b > 0$ one has $\lim_{n\to\infty} J_1 = 0$ because $\frac{a}{a+s} \le \frac{a}{a+b} < 1$ if $s \ge b$. On the other hand, $J_2 \le \int_0^b d(E_s w, w)$, and $\lim_{b\to 0} \int_0^b d(E_s w, w) = 0$ because $w \perp N = E_0 H$. Therefore, given an arbitrary small number $\eta > 0$ one can choose $b > 0$ such that $J_2 \le \eta/2$. Fix this b and choose n sufficiently large so that $J_1 \le \eta/2$. Then $\|B^n w\|^2 \le \eta$. Since η is arbitrarily small, we have proved (7.4) Theorem 9 is proved. $\square$

Remark 4. *Iterative process (7.1) yields a stable solution of equation (1.1). Indeed, let f_δ be given, $\|f_\delta - f\| \le \delta$, and let $u_{n,\delta}$ be defined by (7.1) with f_δ in place of f. Let $w_{n,\delta} = u_{n,\delta} - y$. Then*

$$w_{n_1,\delta} = Bw_{n,\delta} + T_a^{-1} A^*(f_\delta - f), \quad so$$

$$w_{n+1,\delta} = \sum_{j=1}^n B^j T^{-1} A^*(f_\delta - f) + B^n(u_1 - y), \qquad (u_1 - y) \perp N. \tag{7.6}$$

We have proved above that

$$\|B^n(u_1 - y)\| := \mathcal{E}(n) \to 0 \ \ as \ n \to \infty. \tag{7.7}$$

One has:

$$\left\| \sum_{j=0}^n B^j T_a^{-1} A^*(f_\delta - f) \right\| \le \frac{(n+1)\delta}{2\sqrt{a}}, \tag{7.8}$$

because $\|B\| \le 1$ and $\|T_a^{-1} A^*\| \le \frac{1}{2\sqrt{a}}$. From (7.7) and (7.8) one finds the stopping rule, i.e., the number $n(\delta)$ such that $\lim_{\delta\to 0} \|w_{n(\delta),\delta}\| = 0$. This $n(\delta)$ is found for any fixed small δ as the minimizer for the problem

$$\frac{(n+1)\delta}{2\sqrt{a}} + \mathcal{E}(n) = min. \tag{7.9}$$

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Alternatively, one can find $n_1(\delta)$ from the equation

$$\mathcal{E}(n) = \frac{(n+1)\delta}{2\sqrt{a}}. \tag{7.10}$$

Clearly $n(\delta)$ and $n_1(\delta)$ tend to ∞ as $\delta \to 0$.

8 Discrepancy principle for DSM

In this Section we formulate and justify a discrepancy principle for DSM.

Let us start with the version (4.1). We assume that $a(t) > 0$ is a monotonically decaying twice continuously differentiable function, $\lim_{t\to\infty}[a(t) + |\dot{a}| + \ddot{a}] = 0$, $\ddot{a} > 0$, $\lim_{t\to\infty} \frac{\dot{a}(t)}{a(t)} = 0$, for example, $a(t) = \frac{c_1}{(c_0+t)^b}$, where c_1, c_0 and b are positive constants, $b \in (0.5, 1)$. For this $a(t)$ all the assumptions (3.1) hold.

Theorem 10. *The equation*

$$\|AT_{a(t)}^{-1}A^*f_\delta - f_\delta\| = C\delta \qquad C = const, \quad 1 < C < 2, \tag{8.1}$$

has a solution $t = t_\delta$, $\lim_{\delta\to 0} t_\delta = \infty$, such that (4.6) holds, where $u_\delta(t)$ is the solution to (4.1) with f_δ in place of f, $\|f_\delta\| > C\delta$.

Proof of Theorem 10. We have proved earlier that equation (8.1) has a unique solution $a = a_\delta$ and $\lim_{\delta\to 0} a_\delta = 0$. If $a(t)$ is a monotonically decaying function such that $\lim_{t\to\infty} a(t) = 0$, then the equation $a_\delta = a(t)$ defines uniquely $t = t_\delta$, such that $a(t_\delta) = a_\delta$, and $\lim_{\delta\to 0} t_\delta = \infty$.

Let us sketch the proof of relation (4.6), where $u_\delta(t_\delta) = \int_0^{t_\delta} e^{-(t_\delta-s)}T_{a(s)}^{-1}A^*f_\delta ds$ and $t_\delta \to \infty$ as $\delta \to 0$. We have proved (cf (1.17)) that $\|T_{a(t_\delta)}^{-1}A^*f_\delta\| \le \|y\|$, and

$$\lim_{\delta\to 0} \|T_{a(t_\delta)}^{-1}A^*f_\delta - y\| = 0. \tag{8.2}$$

(cf (1.18)). It is clear that $\lim_{t\to\infty} \int_0^t e^{-(t-s)}g(s)ds = g(\infty)$ provided that g is a continuous function and there exists $g(\infty) := \lim_{t\to\infty} g(t)$.

Note that $\lim_{s\to t_\delta} \|T_{a(t_\delta)}^{-1}A^*f_\delta - T_{a(s)}^{-1}A^*f_\delta\| = 0$. We have

$$\int_0^{t_\delta} e^{-(t_\delta-s)}T_{a(s)}^{-1}A^*f_\delta ds = y + o(1) \text{ as } \delta \to 0. \tag{8.3}$$

To check this we use equation (8.2), the formula $T_{a(t_\delta)}^{-1} - T_{a(s)}^{-1} = T_{a(s)}^{-1}[a(t_\delta)-a(s)]T_{a(t_\delta)}^{-1}$, the estimates $\|T_{a(t_\delta)}^{-1}A^*f_\delta\| \le \|y\|$ and $\|T_{a(s)}^{-1}\| \le \frac{1}{a(s)}$, and the relation $\lim_{t\to\infty} \int_0^t e^{-(t-s)}\frac{a(s)-a(t)}{a(s)}ds = 0$, which holds due to our assumptions on $a(t)$. Theorem 10 is proved. $\square$

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